3.1.5 Answers

1. $4x^2 + 3x - 1 = (x - 3)(4x + 15) + 44$

2. $2x^3 - x + 1 = \left(x^2 + x + 1\right)(2x - 2) + (-x + 3)$

3. $5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2 + 4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$

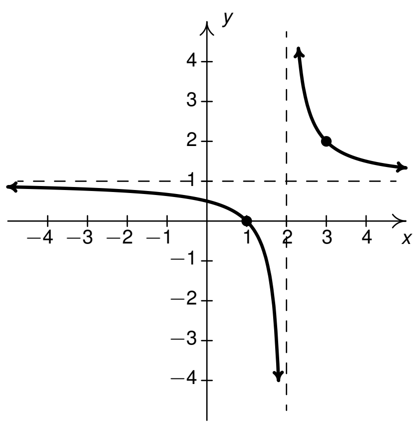
4. $-x^5 + 7x^3 - x = \left(x^3 - x^2 + 1\right)\left(-x^2 - x + 6\right) + \left(7x^2 - 6\right)$

5. $9x^3 + 5 = (2x - 3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

6. $4x^2 - x - 23 = \left(x^2 - 1\right)(4) + (-x - 19)$

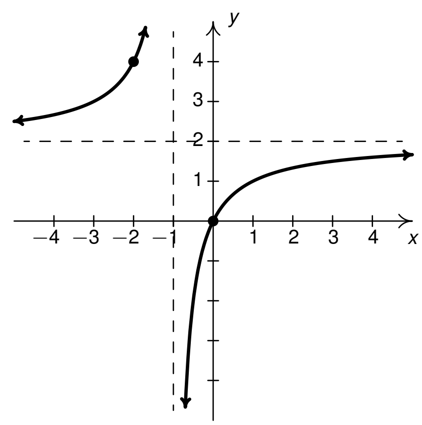
7. $F(x) = \dfrac{1}{x - 2} + 1$

   Domain: $(-\infty, 2) \cup (2, \infty)$
   Range: $(-\infty, 1) \cup (1, \infty)$
   Vertical asymptote: $x = 2$
   Horizontal asymptote: $y = 1$



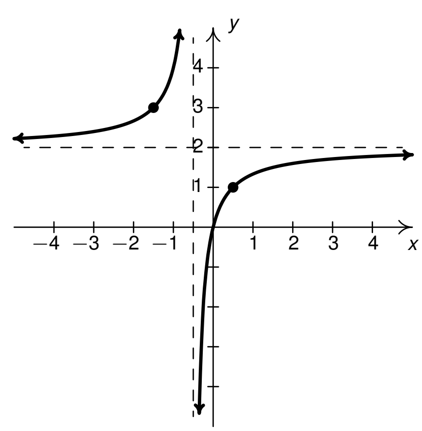
8. $F(x) = \dfrac{2x}{x + 1} = \dfrac{-2}{x + 1} + 2$

   Domain: $(-\infty, -1) \cup (-1, \infty)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -1$
   Horizontal asymptote: $y = 2$



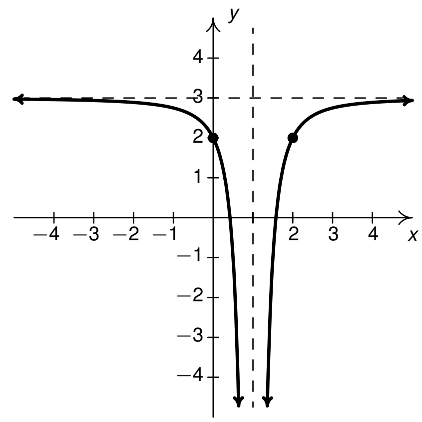
9. $F(x) = 4x(2x + 1)^{-1} = \dfrac{4x}{2x + 1} = \dfrac{-1}{x + \frac{1}{2}} + 2$

   Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -\frac{1}{2}$
   Horizontal asymptote: $y = 2$



10. $F(x) = -(x-1)^{-2} + 3 = \dfrac{-1}{(x-1)^2} + 3$

Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 3$



11. $F(x) = \dfrac{1}{x+2} - 1$

12. $F(x) = \dfrac{-2}{x-1} + 1$

13. $F(x) = \dfrac{-4}{(x+2)^2} + 4$

14. $F(x) = \dfrac{1}{\left(x - \frac{1}{2}\right)^2} - 4$

15. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty$, $f(x) \to \frac{1}{3}^-$
As $x \to \infty$, $f(x) \to \frac{1}{3}^+$

16. $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-$, $f(x) \to \infty$
As $x \to \frac{5}{2}^+$, $f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty$, $f(x) \to -\frac{7}{2}^+$
As $x \to \infty$, $f(x) \to -\frac{7}{2}^-$

17. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4$, $x = 3$
As $x \to -4^-$, $f(x) \to -\infty$
As $x \to -4^+$, $f(x) \to \infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

18. $g(t) = \dfrac{t}{t^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

19. $g(t) = \frac{t+7}{(t+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $t = -3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[21]As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

20. $g(t) = \frac{t^3+1}{t^2-1} = \frac{t^2-t+1}{t-1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $t = 1$
As $t \to 1^-$, $g(t) \to -\infty$
As $t \to 1^+$, $g(t) \to \infty$
Hole at $(-1, -\frac{3}{2})$
Slant asymptote: $y = t$
As $t \to -\infty$, the graph is below $y = t$
As $t \to \infty$, the graph is above $y = t$

21. $r(z) = \frac{4z}{z^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

22. $r(z) = \frac{4z}{z^2-4} = \frac{4z}{(z+2)(z-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
Vertical asymptotes: $z = -2$, $z = 2$
As $z \to -2^-$, $r(z) \to -\infty$
As $z \to -2^+$, $r(z) \to \infty$
As $z \to 2^-$, $r(z) \to -\infty$
As $z \to 2^+$, $r(z) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

23. $r(z) = \frac{z^2-z-12}{z^2+z-6} = \frac{z-4}{z-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $z = 2$
As $z \to 2^-$, $r(z) \to \infty$
As $z \to 2^+$, $r(z) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $z \to -\infty$, $r(z) \to 1^+$
As $z \to \infty$, $r(z) \to 1^-$

24. $f(x) = \frac{3x^2-5x-2}{x^2-9} = \frac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -3$, $x = 3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty$, $f(x) \to 3^+$
As $x \to \infty$, $f(x) \to 3^-$

[21]This is hard to see on the calculator, but trust me, the graph is below the t-axis to the left of $t = -7$.

25. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$

26. $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = x$
As $x \to -\infty$, the graph is above $y = x$
As $x \to \infty$, the graph is below $y = x$

27. $g(t) = \dfrac{2t^2 + 5t - 3}{3t + 2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $t = -\frac{2}{3}$
As $t \to -\frac{2}{3}^-$, $g(t) \to \infty$
As $t \to -\frac{2}{3}^+$, $g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to -\infty$, the graph is above $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to \infty$, the graph is below $y = \frac{2}{3}t + \frac{11}{9}$

28. $g(t) = \dfrac{-t^3 + 4t}{t^2 - 9} = \dfrac{-t^3 + 4t}{(t-3)(t+3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $t = -3$, $t = 3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to -\infty$
As $t \to 3^-$, $g(t) \to \infty$
As $t \to 3^+$, $g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -t$
As $t \to -\infty$, the graph is above $y = -t$
As $t \to \infty$, the graph is below $y = -t$

29. $g(t) = \dfrac{-5t^4 - 3t^3 + t^2 - 10}{t^3 - 3t^2 + 3t - 1} = \dfrac{-5t^4 - 3t^3 + t^2 - 10}{(t-1)^3}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $t = 1$
As $t \to 1^-$, $g(t) \to \infty$
As $t \to 1^+$, $g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -5t - 18$
As $t \to -\infty$, the graph is above $y = -5t - 18$
As $t \to \infty$, the graph is below $y = -5t - 18$

30. $r(z) = \dfrac{z^3}{1 - z}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $z = 1$
As $z \to 1^-$, $r(z) \to \infty$
As $z \to 1^+$, $r(z) \to -\infty$
No holes in the graph
No horizontal or slant asymptote
As $z \to -\infty$, $r(z) \to -\infty$
As $z \to \infty$, $r(z) \to -\infty$

31. $r(z) = \dfrac{18 - 2z^2}{z^2 - 9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
As $z \to \pm\infty$, $r(z) = -2$

32. $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1} = z - 5$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = z - 5$
$r(z) = z - 5$ everywhere.

33. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.

(b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.

(c) For \$40000 you could remove about 95.76% of the fish.

34. (a) $\overline{v}(t) = \frac{s(t)-s(5)}{t-5} = \frac{-5t^2+100t-375}{t-5} = -5t + 75$, $t \neq 5$. The instantaneous velocity of the rocket when $t_0 = 5$ is $-5(5) + 75 = 50$ meaning it is traveling 50 feet per second upwards.

(b) $\overline{v}(t) = \frac{s(t)-s(9)}{t-9} = \frac{-5t^2+100t-495}{t-9} = -5t + 55$, $t \neq 9$. The instantaneous velocity of the rocket when $t_0 = 9$ is $-5(9) + 55 = 10$, so the rocket has slowed to 10 feet per second (but still heading up.)

(c) $\overline{v}(t) = \frac{s(t)-s(10)}{t-10} = \frac{-5t^2+100t-495}{t-10} = -5t + 50$, $t \neq 10$. The instantaneous velocity of the rocket when $t_0 = 10$ is $-5(10) + 50 = 0$, so the rocket has momentarily stopped! In Example 1.2.8, we learned the rocket reaches its maximum height when $t = 10$ seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is 0.

(d) $\overline{v}(t) = \frac{s(t)-s(11)}{t-11} = \frac{-5t^2+100t-495}{t-11} = -5t + 45$, $t \neq 11$. The instantaneous velocity of the rocket when $t_0 = 11$ is $-5(11) + 45 = -10$ meaning the rocket has, indeed, changed direction and is heading downwards at a rate of 10 feet per second. (Note the symmetry here between this answer and our answer when $t = 9$.)

35. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y = 150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

36. (a) $\overline{C}(x) = \frac{100x+2000}{x} = 100 + \frac{2000}{x}$, $x > 0$.

(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.

(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.

(d) As $x \to 0^+$, $\overline{C}(x) \to \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.

(e) As $x \to \infty$, $\overline{C}(x) \to 100^+$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

37. (a) The cost to make 0 items is $C(0) = m(0) + b = b$. Hence, so the fixed costs are b.

(b) $C(x) = mx + b$ is a linear function with slope $m > 0$. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.

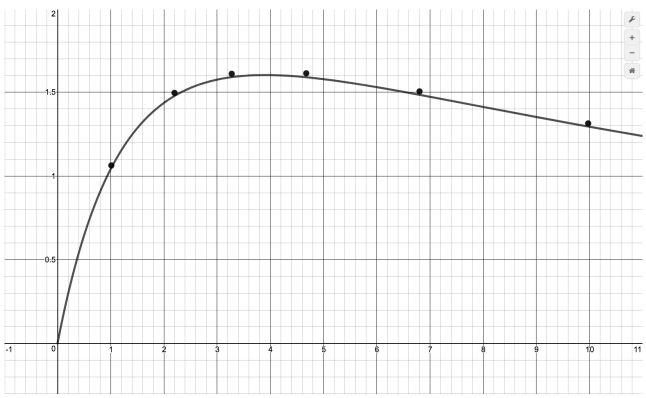
(c) $\overline{C}(x) = \frac{C(x)}{x} = \frac{mx+b}{x} = m + \frac{b}{x}$ for $x > 0$.

(d) Since $b > 0$, $\overline{C}(x) = m + \frac{b}{x} > m$ for $x > 0$. As $x \to \infty$, $\frac{b}{x} \to 0$ so $\overline{C}(x) = m + \frac{b}{x} \to m$.

(e) Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\frac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

38. If $p(x) = mx + b$ and $C(x)$ is linear, say $C(x) = rx + s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx + b) - (rx + s)$ which simplifies to $P(x) = mx^2 + (b - r)x - s$. Hence, the average profit $\overline{P}(x) = \frac{P(x)}{x} = \frac{mx^2+(b-r)x-s}{x} = mx + (b - r) - \frac{s}{x}$. We see that as $x \to \infty$, $\frac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b - r)$. Hence, $y = mx + (b - r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

39. (a)



(b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

(c) As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

40. $a = -2$ and $c = -18$ so $f(x) = \dfrac{-2x^2 + 18}{x + 3}$.

41. (a) $a = 6$ and $n = 2$ so $f(x) = \dfrac{6x^2 - 4}{2x^2 + 1}$ (b) $a = 10$ and $n = 3$ so $f(x) = \dfrac{10x^3 - 4}{2x^2 + 1}$.

42. If we define $f(x) = p(x) - p(a)$ then f is a polynomial function with $f(a) = p(a) - p(a) = 0$. The Factor Theorem guarantees $(x - a)$ is a factor of $f(x)$, that is, $f(x) = p(x) - p(a) = (x - a)q(x)$ for some polynomial $q(x)$. Hence, $r(x) = \frac{p(x)-p(a)}{x-a} = \frac{(x-a)q(x)}{x-a} = q(x)$ so the graph of $y = r(x)$ is the same as the graph of the polynomial $y = q(x)$ except for a hole when $x = a$.

43. The slope of the curves near $x = 1$ matches the exponent on x. This exactly what we saw in Exercise 51 in Section 2.1.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4